The lightlike geometry of spacelike submanifolds in Minkowski space

joint work with

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§1 Gaussian Differential Geometry; brief review

- \( X : U \rightarrow \mathbb{R}^n \): an embedding (hypersurface) \((U \subset \mathbb{R}^{n-1}: \text{open, } X(U) = M.)\)

- \( n(u) \): Unit normal vector at \( p = X(u) \).

- The Gauss map: \( G_M : U \rightarrow S^{n-1} : G_M(u) = n(u) \)

- \( S_p = -dG_M(u) : T_pM \rightarrow T_pM \): the shape operator.

- The Gauss-Kronecker curvature: \( K(p) = \det S_p \).

- The mean curvature: \( H(p) = \frac{1}{n-1} \text{Trace } S_p \).

- Related results on hypersurfaces: The Gauss-Bonnet theorem, The Weierstrass representation formula for a minimal surface, etc.

- For a general submanifolds \( M^s \subset \mathbb{R}^n \), consider the unit normal bundle \( N_1(M) \).

- The (generalized) Gauss map: \( G_{N_1(M)} : N_1(M) \rightarrow S^{n-1} : G_{N_1(M)}(p, \xi) = \xi \)

- The Lipschitz-Killing curvature at \((p, \xi) : K(p, \xi)\) (can be defined).

- The total (absolute) curvature of \( M \) at \( p \): \( K^*(p) = \int_{\xi \in N_1(M)_p} |K(p, \xi)| \text{d}\sigma_{n-s-1} \).

- The related results: The Chern-Lashof theorem, Convexity, Tightness etc.
§2 Lorentz-Minkowski space: $\mathbb{R}^{n+1}_1$

- $\mathbb{R}^{n+1}_1 = (\mathbb{R}^{n+1}, \langle , \rangle)$: Lorentz-Minkowski $n + 1$-space

- $\langle x, y \rangle = -x_0 y_0 + \sum_{i=1}^{n} x_i y_i$, where $x = (x_0, x_1, \ldots, x_n), y = (y_0, y_1 \ldots, y_n)$

- $x \in \mathbb{R}^{n+1}_1 \setminus \{0\}$ is

  \[
  \begin{cases}
  \text{spacelike} & \text{if } \langle x, x \rangle > 0 \\
  \text{lightlike} & \text{if } \langle x, x \rangle = 0 \\
  \text{timelike} & \text{if } \langle x, x \rangle < 0,
  \end{cases}
  \]

- For any lightlike vector $x$, define

  \[
  \tilde{x} = \left(1, \frac{x_1}{x_0}, \ldots, \frac{x_n}{x_0}\right) \in S^{n-1}_+ = \{x = (x_0, x_1, \ldots, x_n) \mid \langle x, x \rangle = 0, x_0 = 1\}.
  \]

- **Hyperplane with pseudo normal** $n$: $HP(n, c) = \{x \in \mathbb{R}^{n+1}_1 | \langle x, n \rangle = c\}$ for any $n \in \mathbb{R}^{n+1}_1 \setminus \{0\}$ and $c \in \mathbb{R}$.

- $HP(n, c)$ is

  \[
  \begin{cases}
  \text{spacelike} & \text{if } n \text{ is timelike} \\
  \text{lightlike} & \text{if } n \text{ is lightlike} \\
  \text{timelike} & \text{if } n \text{ is spacelike},
  \end{cases}
  \]
§2 Lorentz-Minkowski space: $\mathbb{R}^{n+1}_1$

- **Pseudo-spheres** in $\mathbb{R}^{n+1}_1$:
  \[
  \begin{align*}
  H^n(-1) &= \{ x \in \mathbb{R}^{n+1}_1 | \langle x, x \rangle = -1 \} : \text{Hyperbolic n-space} \\
  S^n_1 &= \{ x \in \mathbb{R}^{n+1}_1 | \langle x, x \rangle = 1 \} : \text{de Sitter n-space} \\
  LC^* &= \{ x \neq 0 \in \mathbb{R}^{n+1}_1 | \langle x, x \rangle = 0 \} : \text{(open) lightcone}
  \end{align*}
  \]

- For any $x_1, x_2, \ldots, x_n \in \mathbb{R}^{n+1}_1$,
  \[
  x_1 \wedge x_2 \wedge \cdots \wedge x_n = \begin{vmatrix}
  -e_0 & e_1 & \cdots & e_n \\
  x_1^0 & x_1^1 & \cdots & x_1^n \\
  x_2^0 & x_2^1 & \cdots & x_2^n \\
  \vdots & \vdots & \cdots & \vdots \\
  x_n^0 & x_n^1 & \cdots & x_n^n \\
  \end{vmatrix}, \quad x_i = (x_i^0, x_i^1, \ldots, x_i^n).
  \]

- $x_1 \wedge x_2 \wedge \cdots \wedge x_n$ are pseudo orthogonal to any $x_i$ ($i = 1, \ldots, n$).
- We choose $e_0 = (1, 0, \ldots, 0)$ as the future timelike vector field.
• $X : U \rightarrow \mathbb{R}^{n+1}_1$ : a spacelike embedding ($U \subset \mathbb{R}^{n-1}_1$ : open, $M = X(U)$).
• $T_pM$ : a spacelike subspace at any point $p = X(u) \in M$.
• $N_pM$ : the pseudo-normal space, a timelike plane (i.e., Lorentz plane).
• $N(M)$ : the pseudo-normal bundle over $M$.
• $n^T(u) \in N_p(M)$: arbitrarily future directed unit timelike normal vector field.
• $n^S(u) \in N_p(M)$ : the spacelike unit normal vector field defined by
  \[
  n^S(u) = \frac{n^T(u) \wedge X_{u_1}(u) \wedge \cdots \wedge X_{u_{n-1}}(u)}{||n^T(u) \wedge X_{u_1}(u) \wedge \cdots \wedge X_{u_{n-1}}(u)||}
  \]
• $\{n^T, n^S\}$ : a pseudo-orthonormal frame field along $M$.
• $d(n^T \pm n^S)_u : T_pM \rightarrow T_p\mathbb{R}^{n+1}_1 = T_pM \oplus N_p(M)$ : linear mapping.
• Consider the pseudo-orthogonal projection : $\pi^t : T_pM \oplus N_p(M) \rightarrow T_pM$
• $S_p(n^T, \pm n^S) = -\pi^t \circ d(n^T \pm n^S)^t_u : T_pM \rightarrow T_pM$: $(n^T, \pm n^S)$-shape operator
• The lightcone Gauss-Kronecker curvature with respect to \((n^T, n^S)\):

\[
K^\pm_\ell(n^T, n^S)(u) = \det S_p(n^T, \pm n^S). 
\]

• The lightcone mean curvature with respect to \((n^T, n^S)\):

\[
H^\pm_\ell(n^T, n^S)(u) = \frac{1}{n - 1} \text{Trace } S_p(n^T, \pm n^S). 
\]

• \(\overrightarrow{n}^T(u)\): another future pointed normal vector field; \((n^T \pm n^S) = (\overrightarrow{n}^T \pm \overrightarrow{n}^S) \in S^{n-1}_+\).

• The lightcone Gauss map: \(\overrightarrow{\Pi}^\pm: U \rightarrow S^{n-1}_+ : \overrightarrow{\Pi}^\pm(u) = (n^T \pm n^S)(u)\)

• \(\overrightarrow{S}^\pm_p = -\pi^t \circ d\overrightarrow{\Pi}^\pm_p: T_pM \rightarrow T_pM\): the normalized lightcone shape operator.

• The normalized lightcone Gauss-kronecker curvature: \(\widetilde{K}^\pm_\ell(u) = \det \widetilde{S}^\pm_p\).

• The normalized lightcone mean curvature: \(\widetilde{H}^\pm_\ell(u) = \frac{1}{n - 1} \text{Trace } \widetilde{S}^\pm_p\).
§3 Spacelike submanifolds with codimension two

**Proposition (The relation between curvatures)**

\[
\tilde{K}_\ell^\pm(u) = \left( \frac{1}{\ell_0^\pm(u)} \right)^{n-1} K_\ell^\pm(n^T, n^S)(u), \quad \tilde{H}_\ell^\pm(u) = \left( \frac{1}{\ell_0^\pm(u)} \right) H_\ell^\pm(n^T, n^S)(u),
\]

where \((n^T \pm n^S)(u) = (\ell_0^\pm(u), \ell_1^\pm(u), \ldots, \ell_n^\pm(u))\).

**Corollary**

1. \(\tilde{K}_\ell^\pm(u) = 0\) if and only if \(K_\ell^\pm(n^T, n^S)(u) = 0\).
2. \(\tilde{H}_\ell^\pm(u) = 0\) if and only if \(H_\ell^\pm(n^T, n^S)(u) = 0\).

- The flatness is independent of the choice of \(n^T(u)\).

**Proposition**

Suppose \(n = 3\). Let \(\mathcal{H}\) be the mean curvature vector along \(M\). Then

\[
\tilde{H}_\ell^\pm \equiv 0 \iff \mathcal{H} : \text{lightlike} \iff M : \text{a marginal trapped surface}.
\]
Example

(1) \( n^T \equiv v: \text{constant} \iff M \subset H(v : c): \text{a hypersurface in a spacelike hyperplane} \)

• Special case : \( n^T(u) = e_0 \Rightarrow H(e_0, 0) = \mathbb{R}^n_0: \text{Euclidean space} \)

• \( n^S(u) : \text{the ordinary unit normal in the Euclidean sense.} \)
  \[
  \tilde{K}^\pm(u) = K(u): \text{The Gauss-Kronecker curvature,} \\
  \tilde{H}^\pm(u) = \pm H(u): \text{The mean curvature.} \Rightarrow H \equiv 0: \text{Minimal surfaces} 
  \]

(2) \( n^T(u) = X(u) \Rightarrow M = X(U) \subset H^n(-1): \text{a hypersurface in Hyperbolic space} \)

• \( \tilde{L}^\pm(u) = X(u) \pm n^S(u) : \text{the hyperbolic Gauss map} \)
  \[
  \tilde{K}^\pm(u) = \tilde{K}^\pm_h(u): \text{The horospherical Gauss-Kronecker curvature,} \\
  \tilde{H}^\pm(u) = \tilde{H}^\pm_h(u): \text{The horospherical mean curvature.} \\
  \Rightarrow \tilde{H}^\pm_h \equiv 0: \text{CMC} \pm 1 \text{ surfaces} 
  \]

(3) \( n^S \equiv v: \text{constant} \iff M: \text{a spacelike hypersurface in a timelike hyperplane} \)

\( \Rightarrow \tilde{H}^\pm_\ell \equiv 0: \text{Maximal surfaces} \)

(4) \( n^S(u) = X(u) \Rightarrow M = X(U) \subset S^n_1: \text{a spacelike hypersurface in de Sitter space} \)

• \( \tilde{L}^\pm(u) = n^T(u) \pm X(u) : \text{the de Sitter horospherical Gauss map} \)
§3 Spacelike submanifolds with codimension two

- $M$: closed orientable $(n-1)$-manifold, $f: M \longrightarrow \mathbb{R}^{n+1}_1$: spacelike embedding.
- $\mathbb{R}^{n+1}_1$: time-oriented $\Rightarrow$ globally exists $n^T: M \longrightarrow H^n(-1)$: future directed timelike unit normal vector field along $f(M)$.
- The *global lightcone Gauss map*:
  \[
  \widetilde{L}^\pm: M \longrightarrow S^{n-1}_+: \widetilde{L}^\pm(p) = n^T(p)\pm n^S(p).
  \]
- The *global normalized lightcone Gauss-Kronecker curvature function*:
  \[
  \widetilde{K}^\pm_\ell(p) = \det(-\pi^t \circ d\widetilde{L}_{p}^\pm).
  \]

**Theorem (The Gauss-Bonnet type theorem)**

$M$: a closed orientable, spacelike submanifold of codimension two in $\mathbb{R}^{n+1}_1$.

Suppose that $n$ is odd. Then

\[
\int_M \widetilde{K}_\ell d\nu_M = \frac{1}{2} \gamma_{n-1} \chi(M),
\]

$\chi(M)$: the Euler characteristic of $M$, $d\nu_M$: the volume form of $M$, $\gamma_{n-1}$: the volume of the unit $(n-1)$-sphere $S^{n-1}$. 
§4 Spacelike submanifolds with general codimension

• $X : U \rightarrow \mathbb{R}^{n+1}$: a spacelike embedding of codimension $k$ ($U \subset \mathbb{R}^s$, $M = X(U)$)

• $N_p(M)$: the pseudo-normal space at $p = X(u)$, a $k$-dim Lorentz vector space.

• Two kinds of pseudo spheres:

\[
\begin{align*}
N_p(M; -1) &= \{ v \in N_p(M) \mid \langle v, v \rangle = -1 \} \\
N_p(M; 1) &= \{ v \in N_p(M) \mid \langle v, v \rangle = 1 \}
\end{align*}
\]

• Two unit normal spherical normal bundles over $M$:

\[
N(M; -1) = \bigcup_{p \in M} N_p(M; -1) \text{ and } N(M; 1) = \bigcup_{p \in M} N_p(M; 1).
\]

• Remark that $N_p(M; \pm 1)$ are non-compact $\Rightarrow$ we cannot integrate on the fiber.

• $^3$ future directed unit timelike normal vector field $n^T(p) \in N_p(M; -1)$ (fix!!)

• $N_1(M)_p[n^T] = \{ \xi \in N_p(M; 1) \mid \langle \xi, n^T(p) \rangle = 0 \}$: $k - 1$-spacelike normal sphere.

• $N_1(M)[n^T] = \bigcup_{p \in M} N_1(M)_p[n^T]$: spacelike unit normal bundle w.r.t $n^T$

• Remark that $N_1(M)_p[n^T]$ is compact $\Rightarrow$ we can integrate on the fiber.
The lightcone Gauss map of $N_1(M)[n^T]$:

$$\tilde{LG}(n^T) : N_1(M)[n^T] \to S_+^{n-1} : \tilde{LG}(n^T)(p, \xi) = n^T(p) + \xi$$

- $\Pi^t : \tilde{LG}(n^T)^* TR_{1}^{n+1} = TN_1(M)[n^T] \oplus \mathbb{R}^{k+1} \to TN_1(M)[n^T] :$ the projection.
- $\tilde{S}(n^T)(p, \xi) = -\Pi^t \circ d_{(p, \xi)} \tilde{LG}(n^T) : T_{(p, \xi)} N_1(M)[n^T] \to T_{(p, \xi)} N_1(M)[n^T] :$ the lightcone shape operator.
- $\tilde{K}_{\ell}(n^T)(p, \xi) = \det \tilde{S}(n^T)(p, \xi) :$ the lightcone Lipschitz-Killing curvature of $N_1(M)[n^T]$ at $(p, \xi)$

- Remark: We can apply the theory of Lagarangian/Legendrian singularities to investigate local properties of the lightcone Lipschitz-Killing curvature. However, we do not mention these results here.

Theorem

$$\tilde{LG}(n^T)^* dv_{S_+^{n-1}}(p, \xi) = |\tilde{K}_{\ell}(n^T)(p, \xi)| dv_{N_1(M)[n^T]}(p, \xi),$$

where $dv_{N_1(M)[n^T]} :$ the canonical volume form of $N_1(M)[n^T]$, $dv_{S_+^{n-1}} :$ the canonical volume form of $S_+^{n-1}$. 

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§4 Spacelike submanifolds with general codimension

• ∃ a differential form $d\sigma_{k-2}(n^T)$ of degree $k - 2$ on $N_1(M)[n^T]$ s.t its restriction to a fiber is the volume element of the unit $k - 2$-sphere and

$$dv_{N_1(M)[n^T]} = dv_M \wedge d\sigma_{k-2}(n^T).$$

Proposition (Uniqueness)

Let $n^T, \bar{n}^T$ be future directed unit timelike normal vector fields along $M$. Then we have

$$\int_{N_1(M)_p[n^T]} |\widetilde{K}_t(n^T)(p, \xi)| d\sigma_{k-2}(n^T) = \int_{N_1(M)_p[\bar{n}^T]} |\widetilde{K}_t(\bar{n}^T)(p, \bar{\xi})| d\sigma_{k-2}(\bar{n}^T).$$

• The total absolute lightcone curvature of $M$ at $p$ (well-defined):

$$K^*_t(p) = \int_{N_1(M)_p[n^T]} |\widetilde{K}_t(n^T)(p, \xi)| d\sigma_{k-2}(n^T).$$
§ 4 Spacelike submanifolds with general codimension

- \( f : M \longrightarrow \mathbb{R}^{n+1} \) (\( M \): \( s \)-dim closed orientable manifold): a spacelike immersion
- The total absolute lightcone curvature of \( M \):

\[
\tau_{\ell}(M, f) = \frac{1}{\gamma_{n-1}} \int_M K^*(p)dv_M = \frac{1}{\gamma_{n-1}} \int_{N_1(M)[n^T]} |\widetilde{K}_{\ell}(n^T)(p, \xi)|dv_{N_1(M)[n^T]},
\]

where \( \gamma_{n-1} \) is the volume of the unit \( n-1 \)-sphere \( S^{n-1} \).

Theorem (The Chern-Lashof type theorem)

1. \( \tau_{\ell}(M, f) \geq \gamma(M) \geq 2 \),
2. If \( \tau_{\ell}(M, f) < 3 \), then \( M \) is homeomorphic to the sphere \( S^s \), where \( \gamma(M) \) is the Morse number of \( M \).

- Problem: Suppose \( \tau_{\ell}(M, f) = 2 \).
  What kind of immersed spheres in \( \mathbb{R}^{n+1} \) we have?
- This problem leads the notion of lightlike convexity and lightlike tightness.
§4 Spacelike submanifolds with codimension two, revisited

- If \( s = n - 1 \), then \( N_1(M)[n^T] \) is a double covering of \( M \).
- \( \exists \sigma(p) = (p, n^S(p)) \): global section of \( N_1(M)[n^T] \). \( \Rightarrow K^*(p) = |\widetilde{K}^+(p)| + |\widetilde{K}^-(p)| \).
- The positive/or negative total absolute curvature of \( M \):
  \[
  \tau_{x}^{\pm}(M, f) = \frac{1}{\gamma_{n-1}} \int_M |\widetilde{K}| dv_M.
  \]

Theorem (The strong Chern-Lashof type theorem)

\[
\tau_{\ell}^{\pm}(M, f) \geq 1.
\]

- Remark that \( \exists M \) such that \( \tau_{\ell}^{+}(M, f) \neq \tau_{\ell}^{-}(M, f) \).
- Independent lightlike vectors \( v_i \ (i = 1, 2) \) ⇒ lightlike hyperplanes \( HP(v_i : c_i) \).
- If \( HP(v_1 : c_1) \cap HP(v_2 : c_2) \neq \emptyset \), then \( HP(v_1 : c_1) \cup HP(v_2 : c_2) \) divides \( \mathbb{R}^{n+1}_1 \) into 4 regions. \( ; \) Two timelike regions and two spacelike regions.
- \( f(M) \) is lightlike convex if \( f(M) \) lies entirely in one of the closed half-spacelike regions determined by the tangent lightlike hypersurfaces of \( f(M) \) at any \( p \in M \).

Theorem

\[
\tau_{\ell}^{\pm}(M, f) = 1 \Leftrightarrow M \text{ is homeomorphic to } S^{n-1} \text{ and } f(M) \text{ is lightlike convex}.
\]
Thank you very much for your attention!

And

Happy birthday Reiko and Keizo!