Research on Lagrangian intersections and leaf-wise intersections

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Symplectic manifolds

Definition

\((P^{2n}, \Omega)\) : symplectic manifold
\[\iff\]

1. \(\Omega\) is a closed 2-form on \(P\)
2. \(\Omega\) is nondegenerate

Definition

\(L^n \subset P^{2n}\) is Lagrangian
\[\iff \Omega|_L = 0\]
For any manifold $M$, the cotangent bundle $T^* M$ has the natural symplectic form $\Omega_M$:

$$\Omega_M = \sum_{j=1}^{n} dx_j \wedge dy_j$$

( $x_j$ : coordinates of $M$ , $y_j$ : fiber coordinates )

This symplectic form $\Omega_M$ is called the canonical form on $T^* M$
Let $\phi$ be a 1-form on $M$.

$$L = \phi(M) : \text{Lag} \iff d\phi = 0 \quad (\phi \in Z^1(M))$$

*Figure: horizontal Lagrangian submanifold*
Let $L_1 = \phi_1(M)$, $L_2 = \phi_2(M)$ ($\phi_1, \phi_2 \in Z^1(M)$) be horizontal Lagrangian submanifolds. Define

$$\Gamma := \phi_2 - \phi_1 \in Z^1(M)$$

$$G := \frac{1}{2}(\phi_1 + \phi_2) : M \to T^*M$$

Then,

$$x \in \text{Zero}(\Gamma) \Rightarrow \phi_1(x) = \phi_2(x)$$

$$\Rightarrow G(x) = \phi_1(x) = \phi_2(x) \in L_1 \cap L_2$$

The intersection of two horizontal Lagrangian submanifolds is given by the zero points of closed 1-form $\Gamma$. 
A. Weinstein’s theorem

A. Weinstein extended this property for Lagrangian intersections of general symplectic manifolds.

Theorem (Weinstein, 1973)

\((P, \Omega) : \text{symplectic manifold}\)
\(L_1, L_2 : \text{Lagrangian submanifolds of } P\)
\(\Sigma := L_1 \cap L_2 \text{ is a submanifold of } P\)

\[\Rightarrow \] If a pair of Lagrangian submanifolds \((L'_1, L'_2)\) is \(C^1\) close to \((L_1, L_2)\), there are
\[
\Gamma \in Z^1(\Sigma) \\
G : \Sigma \to P
\]
such that
\[p \in \text{Zero}(\Gamma) \Rightarrow G(p) \in L'_1 \cap L'_2\]
A. Weinstein’s theorem

In particular, consider the case that

- \((P, \Omega) = (T^*M, \Omega_M)\)
- \(L_1 = L_2(=: L)\) : horizontal Lag

Then, \(\Sigma = L \cong M\)

If \(L_1', L_2'\) are \(C^1\) close to \(L\), then \(L_1'\) and \(L_2'\) are also horizontal and the intersection is given by the zero points of some \(\Gamma \in Z^1(M)\).
Consider the topological invariant \( \text{cat}(\Sigma) \), **Lusternik-Schnirelmann category** of \( \Sigma \). This is the least number of contractible open sets which cover \( \Sigma \). Then, the next theorem holds.

**Theorem (Lusternik, Schnirelmann)**

\( \Sigma \) : a compact manifold  
\( f \) : a smooth function on \( \Sigma \)  
\( \Rightarrow \) The number of critical points of \( f \) is at least \( \text{cat}(\Sigma) \)

In the Weinstein’s theorem, if \( \Sigma \) is compact and \( \Gamma = df \), the number of critical points of \( f \) (or zero points of \( \Gamma \)) is at least \( \text{cat}(\Sigma) \). Therefore \( L'_1 \cap L'_2 \) has at least \( \text{cat}(\Sigma) \) points.
Coisotropic submanifolds

Definition

\[ M^{2n-r} \subset P \text{ is a coisotropic submanifold } (0 \leq r \leq n) \iff (T_p M)^\Omega \subset T_p M \ (p \in M) \]

where

\[ (T_p M)^\Omega := \{ v \in T_p P \mid \Omega(v, w) = 0 \ (\forall w \in T_p M) \} \]

For example, in the \(2n\) dimensional Euclidean space with canonical form \((\mathbb{R}^{2n}, \Omega_0)\), define \(M\) as follows

\[ M := \{ (x_1, \cdots, x_n, y_1, \cdots, y_n) \mid y_{n-r+1} = \cdots = y_n = 0 \} \]

Then, \(M\) is a coisotropic submanifold.

\[ M : \text{Lag} \iff (TM)^\Omega = TM \iff M : \text{coisotropic, } r = n \]
Coisotropic submanifolds

For a coisotropic submanifold $M$, $(TM)^\Omega \subset TM$ is a completely integrable distribution in $M$. This defines a foliation of $M$.
Let $L_p$ be the leaf through $p \in M$.

$M^{2n-r} \subset P^{2n}$

Figure: a coisotropic submanifold with leaves

Since

$$\dim T_p M + \dim (T_p M)^\Omega = 2n$$

holds, the dimension of the foliation is $r$. 
Leaf-wise intersections

$M \subset P$ : coisotropic submanifold

$\psi \in \text{Symp}(P) := \{\psi \in \text{Diff}(P) \mid \psi^*\Omega = \Omega\}$

**Definition**

$p \in M$ : leaf-wise intersection of $\psi$

$\iff \psi(p) \in L_p$

**Figure:** A leaf-wise intersection
Leaf-wise intersections

\( \text{LWI}_M(\psi) \) denotes the set of the leaf-wise intersections of \( \psi \):

\[
\text{LWI}_M(\psi) := \{ p \in M \mid \psi(p) \in L_p \}
\]

Examples of leaf-wise intersections:

- Fixed points of \( \psi \in \text{Symp}(P) \) (\( r = 0 \))
- Lagrangian intersections (\( r = n \))
- Periodic orbits in Hamiltonian systems (\( r = 1 \))
Examples of leaf-wise intersections

(1) \( (r = 0) \) First example is the case \( M = P \). In this case,

\[
(T_p M)^\Omega = \{0\}, \quad L_p = \{p\}
\]

Then,

\[
p \in \text{LWI}_M(\psi) \iff \psi(p) = p \iff p \in \text{Fix}(\psi)
\]

In this example, leaf-wise intersections of \( \psi \) are fixed points of \( \psi \).
Examples of leaf-wise intersections

(2) \((r = n)\) Second example is the case of connected Lagrangian submanifold \(M \subset P\).

\[
(T_pM)^\Omega = T_pM, \quad L_p = M
\]

Then,

\[
p \in \text{LWI}_M(\psi) \iff \psi(p) \in M \\
\iff \psi(p) \in M \cap \psi(M)
\]

Since \(\psi\) preserves the symplectic form, \(\psi(M)\) is also a Lagrangian submanifold of \(P\). In this example, leaf-wise intersections of \(\psi\) are Lagrangian intersections of two Lagrangian submanifolds \(M, \psi(M)\).
Examples of leaf-wise intersections

(3) \( r = 1 \) Third example is about periodic orbits of Hamiltonian systems. Let \( (P, \Omega, H) \) be a Hamiltonian system. Define the Hamiltonian vector field \( X_H \) of \( H \) by

\[
X_H \lceil \Omega = dH
\]

The integral curves of \( X_H \) are called orbits of Hamiltonian system \( (P, \Omega, H) \).

Let \( c \in \mathbb{R} \) be a regular value of \( H \) and define

\[
M := H^{-1}(c).
\]

Assume that all orbits in \( M \) are periodic.

For example, consider \( (P, \Omega) = (\mathbb{R}^{2n}, \Omega_0) \) and the Hamiltonian function

\[
H = \frac{\alpha}{2} \sum_{K=1}^{n} \lambda (x_k^2 + y_k^2) \ (\alpha > 0, \ \lambda_k \in \mathbb{N})
\]

then, all orbits in \( M = H^{-1}(c) \ (c > 0) \) are periodic.
Examples of leaf-wise intersections

Consider the perturbation $H + \tilde{H}$ of this Hamiltonian system. Here,

$$\tilde{H} : P \times \mathbb{R} \to \mathbb{R}, \quad \tilde{H}(\cdot, t) \equiv 0 \ (t \leq a, \ b \leq t)$$

**Figure:** definition of $\psi$
$M = H^{-1}(c)$ is coisotropic and the leaves of $M$ are orbits of $H$. If $p \in \text{LWI}_M(\psi)$, $\psi(p)$ lies on the orbit of $H$ through $p$. Then, the orbit of the perturbed system $(P, \Omega, H + \tilde{H})$ through $p \in \text{LWI}_M(\psi)$ at $t = a$ is periodic.

In this example, the leaf-wise intersections of $\psi$ are the periodic orbits.
J. Moser proved the next theorem about leaf-wise intersections.

**Theorem (Moser, 1978)**

\[(P, \Omega) : \text{simply connected symplectic manifold}\]
\[\Omega : \text{exact form}\]
\[M \subset P : \text{compact coisotropic submanifold}\]
\[\Rightarrow\]

If \(\psi \in \text{Symp}(P)\) is \(C^1\) close to \(id_P : P \to P\), \(\psi\) has at least two leaf-wise intersections.
I proved the next theorem by using the Weinstein’s theorem.

**Main Theorem**

\[(P, \Omega) : \text{symplectic manifold} \]
\[M \subset P : \text{coisotropic submanifold} \]
\[\Rightarrow \]

If \( \psi \in \text{Symp}(P) \) is \( C^1 \) close to \( \text{id}_P : P \rightarrow P \), there are

\[ \Gamma \in Z^1(M) \]
\[ G : M \rightarrow P \times P \]

such that

\[ p \in \text{Zero}(\Gamma) \Rightarrow \pi_1 \circ G(p) \in \text{LWI}_M(\psi) \]

Particularly, if \( M \) is compact and \( \Gamma = df \), then \( \psi \) has at least \( \text{cat}(\Sigma) \) leaf-wise intersections.
In this theorem, suppose that $P$ is simply connected. Then $\Gamma$ is exact. Moreover, suppose that $M$ is compact. Then any smooth function on $M$ has at least two critical points. Therefore, $\psi$ has at least two leaf-wise intersections.

Hence,

Main Theorem $\Rightarrow$ Theorem (Moser)
Remarks on main theorem

There is a LS category for foliated manifolds. We only have the evaluation of the critical points of a function which is constant on each leaves by this LS category.

On the other hand, the function $f$ on main theorem is not necessarily constant on each leaves.

The extended LS category is useless for evaluating the number of leaf-wise intersections of a symplectomorphism $\psi$. 
Thank you very much!